THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MATH1010 I/J University Mathematics 2015-2016
Problem Set 1

1. Sketch the graphs of the following functions.

   (a) $f : \mathbb{R} \to \mathbb{R}$ defined by
   $$f(x) = \begin{cases} 1 & \text{if} \quad x = 0, \\ 0 & \text{if} \quad x \neq 0. \end{cases}$$

   (b) $f : \mathbb{R} \to \mathbb{R}$ defined by
   $$f(x) = \begin{cases} 1 & \text{if} \quad x > 0, \\ 0 & \text{if} \quad x = 0, \\ -1 & \text{if} \quad x < 0. \end{cases}$$

   (c) $f : \mathbb{R}\backslash\{1\} \to \mathbb{R}$ defined by $f(x) = \dfrac{x^2 - 1}{x - 1}$.

   (d) $f : [-1, 1] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$ defined by $f(x) = \sin^{-1} x$.

   (e) $f : [-1, 1] \to [0, \pi]$ defined by $f(x) = \cos^{-1} x$.

   (f) $f : \mathbb{R} \to (-\frac{\pi}{2}, \frac{\pi}{2})$ defined by $f(x) = \tan^{-1} x$.

2. Prove that $1 \times 2 + 2 \times 3 + 3 \times 4 + \cdots + n \times (n + 1) = \dfrac{n(n + 1)(n + 2)}{3}$ for all natural numbers n.

3. Prove that $8^n - 3^n$ is divisible by 5 for all natural numbers n.

4. A sequence $\{a_n\}$ is defined by
   $$a_1 = 3 \text{ and } a_{n+1} = \sqrt{a_n + 5} \text{ for } n \geq 1.$$
   Show that $a_n \geq a_{n+1}$ for all natural numbers n.

5. A sequence $\{a_n\}$ is defined by
   $$a_1 = 4 \text{ and } a_{n+1} = \frac{6(a_n^2 + 1)}{a_n^2 + 11} \text{ for } n \geq 1.$$

   (a) Show that $a_n > 3$ for all natural numbers n.

   (b) Show that $a_n \geq a_{n+1}$ for all natural numbers n.

6. Show that $\cos 20^\circ \cdot \cos 40^\circ \cdot \cos 80^\circ = \dfrac{1}{8}$.

7. If $\sin(A + B) = 3 \sin(A - B)$, prove that $\tan A = 2 \tan B$.

8. Prove that $\dfrac{\cos(A + B) + \cos(A - B)}{\sin(A - B) - \sin(A + B)} = -\cot B$.

9. (a) Prove that $\dfrac{\sin 5A}{\sin A} - \dfrac{\cos 5A}{\cos A} = 4 \cos 2A$.

(b) Hence, solve the equation $\dfrac{\sin 5A}{\sin A} - \dfrac{\cos 5A}{\cos A} = 2$ where $0° < A < 180°$ and $A \neq 90°$.

10. Prove the triple angle formulas:

    (a) $\sin 3A = 3\sin A - 4\sin^3 A$;

    (b) $\cos 3A = 4\cos^3 A - 3\cos A$;

    (c) $\tan 3A = \dfrac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$.

11. If $A + B + C = 180°$, prove that

    (a) $\sin 2A + \sin 2B + \sin 2C = 4\sin A \sin B \sin C$;

    (b) $\tan A + \tan B + \tan C = \tan A \tan B \tan C$;

    (c) $\cot A \cot B + \cot B \cot C + \cot C \cot A = 1$.